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ON ⊕-δ- SUPPLEMENTED MODULES

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Abstract

Let R be a ring and M a right R-module. M is called ⊕-δ- supplemented, if every submodule of M has a δ- supplemented that is a direct summand of M. In this paper, several properties of these modules are studied. Also, we investigate some properties of rings whose modules are ⊕-δ- supplemented.

1. Introduction

Throughout this paper R is an associative ring with unity and all modules are unitary right R-modules. A submodule K of a module M is denoted by $K \leq M$. Let M be any R-module and S a submodule of M. S is called a *small* submodule of M (denoted by $S \ll M$), if for every submodule T of M with $M = S + T$, then $M = T$. Let M be an R-module and N a submodule of M. If any submodule K of M is minimal with the property that $M = N + K$, then the submodule K is called a *supplement* of N in M. K is a supplement of N in M if and only if $M = N + K$ and

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$N \cap K \ll K$. Any R-module M is called *supplemented* if every submodule of M has a supplement in M. The module M is said to be a *lifting* module if for any submodule N of M there exists $A \leq N$ such that $M = A \oplus B$ and $N \cap B \ll B$. As a generalization of small submodules, a submodule N of M is called δ- small in M, denoted by $N \ll_{\delta} M$, if $M = N + K$ with M / K singular implies $M = K$. Singular and nonsingular modules are studied in [3]. The sum of all δ- small submodules of a module M is denoted by $\delta(M)$, which defines a preradical on the category of R-modules, $\delta(M) = \sum\{L \leq M \mid L \ll_{\delta} M\}$ (See [8]). A module M is called δ- supplemented, if for any $N \leq M$, there exists a submodule X of M such that $M = N + X$ with $N \cap X \ll_{\delta} X$. Mohammed and Muller [6] called an R-module M is $\oplus$- supplemented, if every submodule of M has a supplement, that is, a direct summand of M. Clearly, $\oplus$- supplemented modules are supplemented, but the converse is false, See [5, A.4(2)]. A module M is completely $\oplus$- supplemented, if every direct summand of M is $\oplus$- supplemented. A module M is called δ- lifting, if for any $N \leq M$, there exists a decomposition $M = A \oplus B$ such that $A \leq N$ and $N \cap B$ is δ- small in M. An epimorphism $f : P \to N$ is called a δ- cover of N, if $Ker(f) \ll_{\delta} P$ and, if moreover, P is projective, then it is called a projective δ- cover. In [8], a ring R is called δ- semiperfect, if every simple R-module has a projective δ- cover and a ring R is called δ- perfect, if every R-module has a projective δ- cover. A submodule $N \leq M$ is called *cofinite* in M, if the factor module M / N is finitely generated. A module M is called $\oplus$- cofinitely δ- supplemented, if any cofinite submodule of M has a δ- supplement, that is, a direct summand of M. $\oplus$-δ- supplemented modules are $\oplus$- cofinitely δ- supplemented. The module $M_i (1 \leq i \leq n)$ are called relatively projective, if M_i is M_j- projective for all $1 \leq i \neq j \leq n$.

2. Some Properties of ⊕-δ- supplemented

Lemma 2.1. *Let N and L be submodules of a module M such that* $N + L$ *has a* δ- *supplement* H *in* M *and* $N \cap (H + L)$ *has a* δ- *supplement G in N. Then* $H + G$ *is a* δ- *supplement of L in M.*

Proof. Let H be a δ- supplement of $N + L$ in M and let G be a δ- supplement of $N \cap (H + L)$ in N. Then $M = (N + L) + H$ such that $(N + L) \cap H$ is δ- small in H and $N = [N \cap (H + L)] + G$ such that $(H + L) + G$ is δ- small in G. Since $(H + G) \cap L \leq [(G + L) \cap H] + [(H + L) \cap G]$, $H + G$ is a δ- supplement of L in M. □

Theorem 2.2. *For any ring R, any finite direct sum of ⊕-δ- supplemented R- modules is ⊕-δ- supplemented.*

Proof. Let n be any positive integer and for each $(1 \leq i \leq n) M_i$ a ⊕-δ- supplemented. Let $M = M_1 \oplus \ldots \oplus M_n$. It is enough to show that M is ⊕-δ- supplemented when $n = 2$.

Let L be any submodule of M. Then $M = M_1 + M_2 + L$. So that $M_1 + M_2 + L$ has a δ- supplemented 0 in M. Let H be a δ- supplemented of $M_2 \cap (M_1 + L)$ in M_2 such that H is a direct summand of M_2. By Lemma 2.1, H is a δ- supplemented of $M_1 + L$ in M. Let K be a δ- supplemented of $M_1 \cap (L + H)$ in M_1 such that K is a direct summand of M_1. Again by Lemma 2.1, we have that $H + K$ is a δ- supplemented of L in M. Since H is a direct summand of M_2 and K is a direct summand of M_1, it follows that $H + K = H \oplus K$ is a direct summand of M. Thus $M = M_1 \oplus M_2$ is ⊕-δ- supplemented. □

Corollary 2.3. *Any finite direct sum of* δ- *lifting is* ⊕-δ- *supplemented.*

It is easy to see that, under the given definitions, the following implications hold:

δ-*lifting* $\Rightarrow$ $\oplus$-δ-*supplemented* $\Rightarrow$ δ-*supplemented* $\Rightarrow$ *weak*-δ-*supplemented*.

Theorem 2.4. *Let* M_i $(1 \leq i \leq n)$ *be any finite collection of relatively projective modules. Then the module* $M = M_1 \oplus \ldots \oplus M_n$ *is* $\oplus$-δ-*supplemented if and only if* M_i *is* $\oplus$-δ-*supplemented for each* $(1 \leq i \leq n)$.

Proof. The sufficiency is proved in Theorem 2.2. Conversely, we only prove M_1 is $\oplus$-δ-supplemented. Let $A \leq M_1$. There exists $B \leq M$ such that $M = A + B$, where B is a direct summand of M and $A \cap B$ is δ-small in B. Since $M = A + B = M_1 + B$, there exists $B_1 \leq B$ such that $M = M_1 \oplus B_1$, See [5, 4.47]. Then $B = B_1 \oplus (M_1 \cap B)$. Note that $M_1 = A + (M_1 \cap B)$ and $M_1 \cap B$ is a direct summand of M_1. Then $A \cap B = A \cap (M_1 \cap B)$ is δ-small in $M_1 \cap B$. Hence M_1 is $\oplus$-δ-supplemented. □

Let M be any module. M is called a (D_3)-module, if whenever, M_1 and M_2 are direct summands of M with $M = M_1 + M_2$, $M_1 \cap M_2$ is also a direct summand of M.

Theorem 2.5. *Let* M *be a* $\oplus$-δ-*supplemented module with* (D_3). *Then* M *is completely* $\oplus$-δ-*supplemented*.

Proof. Let N be a direct summand of M and A a submodule of N. We show that A has a δ-supplement in N, that is, a direct summand of N. Since M is $\oplus$-δ-supplemented, there exists a direct summand B of M such that $M = A + B$ and $A \cap B$ is δ-small in B. Hence $N = A + (N \cap B)$. Furthermore $N \cap B$ is a direct summand of M because M has (D_3). Then $A \cap (N \cap B) = A \cap B$ is δ-small in $N \cap B$. □

Lemma 2.6. *Let* N *be a direct summand of a module* M *and let* K *be a submodule of* M *such that* M / K *is projective and* $M = N + K$. *Then* $N \cap K$ *is a direct summand of* M.

Proof. See [4, Lemma 2.3]. □

Theorem 2.7. *Let M be a ⊕-δ-supplemented module. Let N be a submodule of M such that M/N is projective. Then N is ⊕-δ-supplemented.*

Proof. Let L be a submodule of N. Since M is ⊕-δ-supplemented, there exists a direct summand K of M such that $M = L + K$ and $L \cap K$ is δ-small in K. Therefore $N = L + (N \cap K)$ and $L \cap (N \cap K) = L \cap K \ll_\delta M$, $L \cap K \ll_\delta N$. By Lemma 2.6, $N \cap K$ is a direct summand of N. Thus N is ⊕-δ-supplemented. □

Let M be a ⊕-δ-supplemented module. We will answer the following natural question: is any factor module of M ⊕-δ-supplemented?

Lemma 2.8. *Let M be a nonzero module and U be a submodule of M such that $f(U) \leq U$ for each $f \in End_R(M)$. If $M = M_1 \oplus M_2$, then $U = (U \cap M_1) \oplus (U \cap M_2)$.*

Proof. See [6, Lemma 2.4]. □

Theorem 2.9. *Let M be a nonzero module and U a submodule of M such that $f(U) \leq U$ for each $f \in End_R(M)$. If M is ⊕-δ-supplemented, then M/U is ⊕-δ-supplemented. If moreover, U is a direct summand of M, then U is also ⊕-δ-supplemented.*

Proof. Suppose that M is ⊕-δ-supplemented. Let L be a submodule of M which contains the submodule U. There exist submodules N and N' of M such that $M = N \oplus N'$, $M = L + N$, and $L \cap N$ is δ-small in N. By ([9], Lemma 1.2(d)), $(N + U)/U$ is a δ-supplement of L/U in M/U. Now apply Lemma 2.5 to get that $U = (U \cap N) \oplus (U \cap N')$.

Thus, $(N+U) \cap (N'+U) \leq (N+U+N') \cap U + (N+U+U) \cap N'$. Hence, $(N + U) \cap (N' + U) \leq U + (N + U \cap N + U \cap N') \cap N'$. It follows that $(N+U) \cap (N'+U) \leq U$ and $((N+U)/U) \oplus ((N'+U)/U) = M/U$. Then $(N+U)/U$ is a direct summand of M/U. Consequently, M/U is ⊕-δ-supplemented. □

Let M be a module. Then M is called distributive, if $N \bigcap (L + K) = (N \bigcap L) + (N \bigcap K)$ and $N + (L \bigcap K) = (N + L) \bigcap (N + K)$ for some submodules N, K, L of M.

Corollary 2.10. *Every factor module of a distributive* $\oplus$-δ-*supplemented module is* $\oplus$-δ- *supplemented.*

3. Rings Whose Modules are $\oplus$-δ- Supplemented

In this section, we study some rings whose modules are $\oplus$-δ-supplemented. We show that every finitely generated right R-module is $\oplus$-δ- supplemented, if and only if every cyclic right R-module is $\oplus$-δ-supplemented and every finitely generated right R-module is a direct sum of cyclic modules. We investigated relation δ- semiperfect with $\oplus$-δ-supplemented modules.

Theorem 3.1. *Let R be any ring and let M be a finitely generated R-module such that every direct summand of M is* $\oplus$-δ- *supplemented. Then M is a direct sum of cyclic modules.*

Proof. Suppose that $M = m_1R + \ldots + m_kR$ for some positive integer k and elements $m_i \in M(1 \leq i \leq n)$. If $k = 1$, then there is nothing to prove. Suppose that $k > 1$ and that the result holds for $(k-1)$- *generated* modules with the stated condition. There exist submodules K, K' of M such that $M = K \oplus K'$, $M = m_1R + K$, and $m_1R \bigcap K \ll_\delta K$. Note that $K' \cong (M / K) = (m_1R + K) / K \cong m_1R / (m_1R \bigcap K)$. so K' is cyclic. On the other hand, $K / (m_1R \bigcap K) \cong (m_1R + K) / m_1R = M / m_1 \quad R$, so $K / (m_1R \bigcap K)$ is $(k-1)$-*generated*. Since $m_1R \bigcap K \ll_\delta K$, it follows that K is $(k-1)$-*generated*. By induction, K is a direct sum of cyclic modules. Thus $M = K \oplus K'$ is a direct sum of cyclic modules. □

Corollary 3.2. *Let R be a ring. Then every* 2-*generated* $\oplus$-δ-*supplemented R-module is a direct sum of cyclic modules.*

Corollary 3.3. *Let R be a ring and n be a positive integer. Then every n-generated R-module is $\oplus$-δ- supplemented if and only if*

(1) *every cyclic R-module is $\oplus$-δ- supplemented, and*

(2) *every n-generated R-module is a direct sum of cyclic modules.*

Proof.

($\Leftarrow$) By Theorem 2.4.

($\Rightarrow$) By Theorem 3.1, since every direct summand of an n-generated module is n-generated. □

Theorem 3.4. *Let R be a ring. Then the R-module R_R is $\oplus$-δ- supplemented, if and only if every finitely generated free R-module is $\oplus$-δ- supplemented.*

Proof. ($\Rightarrow$) Let M be a free R-module and $A = \{a_i | i \in I\}, |I| < \infty$ be a basis of M. By assumption, $a_i R \cong R(i \in I)$ is $\oplus$-δ- supplemented and M is $\oplus$-δ- supplemented by Theorem 2.2.

($\Leftarrow$) The proof is simple. □

Lemma 3.5. *For a ring R the following statements are equivalent:*

(1) *R is δ- semiperfect;*

(2) *R_R is $\oplus$- cofinitely δ- supplemented;*

(3) *Every free R-module is $\oplus$- cofinitely δ- supplemented;*

(4) *Every finitely generated free R-module is $\oplus$-δ- supplemented;*

(5) *R_R is $\oplus$-δ- supplemented.*

Proof.

(1) $\Leftrightarrow$ (2) See [1, Theorem 3.9].

(2) $\Leftrightarrow$ (3) It follows from Lemma 3.5.

(3) $\Leftrightarrow$ (4) Let M be a finitely generated free R-module. By assumption, M is $\oplus$- cofinitely δ- supplemented. Since each submodule N

of M is cofinite in M, N has a δ- supplement which is a direct summand of M. Hence, M is $\oplus$- cofinitely δ- supplemented.

(4) $\Rightarrow$ (5) and (5) $\Rightarrow$ (1) are clear. □

Let R be a δ- semiperfect. Then by Lemma 3.5, R_R is $\oplus$-δ- supplemented and $\oplus$- cofinitely δ- supplemented.

Corollary 3.6. *Let R be a division ring. Then R is δ- semiperfect, if and only if every finitely generated R-module is $\oplus$-δ- supplemented.*

Proof. ($\Leftarrow$) The sufficiency is proved in Lemma 3.5.

($\Rightarrow$) Let R be a δ- semiperfect division ring. Then by [7, 20.10] every R-module is free. Hence, by Lemma 3.5, every R-module is $\oplus$-δ- supplemented. □

Theorem 3.7. *Let R be a ring and M be an R-module such that $M = \oplus_{i\in I} M_i$, where M_i is a δ- lifting module for each $i \in I$, $\delta(M) \ll_\delta M$ and $M/\delta(M)$ is singular. Then M is $\oplus$-δ- supplemented.*

Proof. Let N be a submodule of M. For each $i \in I$, let $J_i = \delta(M_i)$. If $T = \delta(M)$, then $T = \oplus_{i\in I} J_i$ [8, Lemma 1.5], for each $i \in I$, $J_i = T \cap M_i$ and hence $M_i / J_i \cong (M_i + T)/T$ and so is semisimple. Now $M/T = \sum_{i\in I}(M_i + T)/T$. By [2, Lemma 9.2], $M/T = ((N+T)/T) \oplus \{ \oplus_{\alpha\in\Lambda} ((L_\alpha + T)/T)\}$ for some submodule L_α of $M_\alpha (\alpha \in \Lambda)$ and an index set $\Lambda \subseteq I$. Since M_α is δ- lifting, $L_\alpha / K_\alpha \ll_\delta M_\alpha / K_\alpha$, $L_\alpha / K_\alpha \subseteq \delta(M$ $L_\alpha / K_\alpha \ll_\delta M_\alpha / K_\alpha$, $L_\alpha / K_\alpha \subseteq \delta(M_\alpha / K_\alpha) = (J_\alpha + K_\alpha)/K_\alpha$. So, $K_\alpha \subseteq L_\alpha \subseteq J_\alpha + K_\alpha$. $K = \oplus_{\alpha\in\Lambda} K_\alpha$. Then K is a direct summand of M. Note that $M = N + (\sum_{\alpha\in\Lambda} L_\alpha) + T \subseteq N + K + T$, so that $M = N + K + T$ and hence $M = N + K$, since $\delta(M) \ll_\delta M$. Next, $N \cap K \subseteq (N+T) \cap (\sum_{\alpha\in\Lambda} L_\alpha + T) \subseteq T \ll_\delta M$. It follows that $N \cap K \ll_\delta K$. Therefore, M is $\oplus$-δ- supplemented. □

Corollary 3.8. *Let $R / Soc(R)$ be a right perfect ring and let M be an R-module such that $M = \oplus_{i \in I} M_i$, where M_i is δ- lifting for each $i \in I$ and $M / \delta(M)$ is singular. Then M is ⊕-δ- supplemented.*

Proof. By [8, Lemma 3.8], if $R / Soc(R)$ is right perfect ring, then $R / Soc(R)$ is semisimple and $\delta(M) \ll_{\delta} M$ for every module *M*. □

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